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Divergent IR gluon propagator from Ward-Slavnov-Taylor identities?

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Abstract: We exploit the Ward-Slavnov-Taylor identity relating in genaral covariant gauges the 3-gluons to the ghost-gluon vertices to conclude either that the ghost dressing function is finite and non vanishing at zero momentum while the gluon propagator diverges (although it may do so weakly enough not to be in contradiction with current lattice data) or that the 3-gluons vertex is non-regular when one momentum goes to zero. We stress that those results should be kept in mind when one studies the Infrared properties of the ghost and gluon propagators, for example by means of Dyson-Schwinger equations.

Keywords: Lattice QCD, QCD.

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1. Introduction

The infrared behaviour of QCD Green functions is a very active and hot subject. This is particularly true in the Yang-Mills sector. Dyson-Schwinger (DS) equations have been intensively studied but *the consequences of the Ward-Slavnov-Taylor identities (WSTI) have been largely overlooked*. It turns out that, after some regularity assumptions are made on the ghost and gluon vertex functions, *they do provide extremely strong constraints on the zero momentum limit of both the ghost and the gluon propagator*, namely that the ghost dressing function is finite non vanishing and the gluon propagator diverges.[1] The derivation of these results is indeed rather simple and is the main goal of this letter.

We have already presented in previous publications ([1, 2]) arguments in favor of a finite non-vanishing ghost dressing function at zero momentum. Notice that this was partly based on the study of Dyson-Schwinger equations. Now, although the Dyson Schwinger equations are exact, their practical implementation always involves various approximations and hypotheses which cast a doubt on the general validity of the results so obtained. On the contrary, using the WSTI appears to be quite simple and to require only a minimum amount of extra information on the regularity of the vertex functions. In our opinion this circumstance puts its consequences on a very firm ground and we think any acceptable solution for the propagators should comply with them. As shown in ref. [1], a non-vanishing ghost dressing function could be also favored by the analysis of the Ghost propagator Dyson-Schwinger equation. Thus, one is led to conclude either that the gluon propagator diverges and the ghost dressing is finite non-vanishing or that the regularity hypotheses on the vertices should fail.

[1] This divergence can however be so soft as not to contradict the apparent finiteness previously stated from lattice data.

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2. Ward-Slavnov-Taylor identity and the infrared behaviour of the gluon propagator

The ghost-gluon vertex $\widetilde{\Gamma}_\mu$ is written as

$$\widetilde{\Gamma}^{abc}_\mu(p,q;r) = g_0 f^{abc}\,\widetilde{\Gamma}_\mu(p,q;r) = g_0(-ip_\nu) f^{abc}\widetilde{\Gamma}_{\nu\mu}(p,q;r), \tag{2.1}$$

where $-p$ (q) is the outgoing (incoming) ghost momentum and r is that of the gluon, its tensorial structure defined by the following general decomposition [3]:

$$\begin{aligned}\widetilde{\Gamma}_{\nu\mu}(p,q;r) = {}& \delta_{\nu\mu}\; a(p,q;r) \;-\; r_\nu q_\mu\; b(p,q;r) \;+\; p_\nu r_\mu\; c(p,q;r) \\ & + r_\nu p_\mu\; d(p,q;r) \;+\; p_\nu p_\mu\; e(p,q;r)\ .\end{aligned} \tag{2.2}$$

The ghost-gluon vertex is related to the 3-gluon vertex, $\Gamma^{abc}_{\lambda\mu\nu}(p,q,r)$, through the Ward-Slavnov-Taylor identity ([4, 5]):

$$\begin{aligned}p_\lambda \Gamma_{\lambda\mu\nu}(p,q,r) = {}& \frac{F(p^2)}{G(r^2)}(\delta_{\rho\nu} r^2 - r_\rho r_\nu)\widetilde{\Gamma}_{\rho\mu}(r,p;q) \\ & - \frac{F(p^2)}{G(q^2)}(\delta_{\rho\mu} q^2 - q_\rho q_\mu)\widetilde{\Gamma}_{\rho\nu}(q,p;r)\,.\end{aligned} \tag{2.3}$$

In this equation F and G are the ghost and gluon dressing functions, defined respectively as

$$\begin{aligned}\langle c\bar{c}\rangle &= \frac{F(q^2)}{q^2} \qquad\qquad \text{and} \\ \langle A_\mu A_\nu\rangle &= \frac{1}{q^2}\left[G(q^2)\,(\delta_{\mu\nu} - \frac{q_\mu\, q_\nu}{q^2}) + (1-\xi)\frac{q_\mu\, q_\nu}{q^2}\right]\,,\end{aligned} \tag{2.4}$$

with ξ the usual gauge fixing parameter. We recall in this respect that the WSTI holds in any covariant gauge and that the longitudinal part of the propagator is trivial. We now make the hypothesis that $\Gamma_{\lambda\mu\nu}(p,q,r)$ [2] has a well defined limit when anyone of its arguments goes to 0, the other ones being kept non-vanishing. Note that this restricts but does not forbid the possible presence of singularities in the coefficient functions since they could be compensated by kinematical zeroes stemming from the basis tensors. Indeed, this assumption and WSTI are all one needs to conclude that the gluon propagator diverges at zero momentum. The two transversal projectors in the r.h.s. of eq. (2.3) imply that a well defined limit at zero momentum for the l.h.s, after contraction with r_ν or q_μ, can only be zero. For instance, multiplying eq. (2.3) with r_ν gives

$$p_\lambda r_\nu \Gamma_{\lambda\mu\nu}(p,q,r) = -F(p^2)\frac{q^2}{G(q^2)}\left(\delta_{\rho\mu} - \frac{q_\rho q_\mu}{q^2}\right) r_\nu \widetilde{\Gamma}_{\rho\nu}(q,p;r)\ . \tag{2.5}$$

[2]Actually this assumption regards only the *longitudinal part* , *i.e.* with at least one longitudinal gluon, of the vertex function (see ref. [3] for its definition) since the purely transverse one trivially disappears from the STI.

Since we suppose that the left hand side has a well defined limit when q goes to zero, the same has to be true for the right hand side. However, the vector

$$r_\nu \widetilde{\Gamma}_{\rho\nu}(q,p;r) \;\equiv\; X(q,p;r)\; r_\rho + Y(q,p;r)\; q_\rho\,, \tag{2.6}$$

where X and Y are combinations of the form factors introduced earlier in eq. (2.2),

$$\begin{aligned} X(q,p;r) &= a(q,p;r) - (r\cdot p)\; b(q,p;r) + (r\cdot q)\; d(q,p;r) \\ Y(q,p;r) &= r^2 c(q,p;r) + (r\cdot q)\; e(q,p;r)\,, \end{aligned} \tag{2.7}$$

after contraction with the tensor $\delta_{\rho\mu} - \frac{q_\rho q_\mu}{q^2}$,

$$\left(\delta_{\rho\mu} - \frac{q_\rho q_\mu}{q^2}\right) r_\nu \widetilde{\Gamma}_{\rho\nu}(q,p;r) \;=\; \left(r_\mu - \frac{(q\cdot r)}{q^2} q_\mu\right) X(q,p;r) \tag{2.8}$$

gives an explicit dependence on the direction of q. This is in contradiction with a well defined limit unless the zero-momentum limit of both sides in eq. (2.5) is 0.

It is worth noticing that the same conclusion can be otherwise proven by exploiting the following general property of the 3-gluon vertex:

$$p_\lambda q_\mu r_\nu \Gamma_{\lambda\mu\nu}(p,q,r) = 0\;. \tag{2.9}$$

This last result can be straightforwardly derived from WSTI, and it is supported by the perturbative results for the 3-gluon vertex in ref. [6]. Then, as $-p = q + r$

$$q_\lambda q_\mu r_\nu \Gamma_{\lambda\mu\nu}(p,q,r) + r_\lambda q_\mu r_\nu \Gamma_{\lambda\mu\nu}(p,q,r) = 0\;. \tag{2.10}$$

Thus, simply by considering the leading behaviour as $q \to 0$ one proves that:

$$\lim_{q\to 0} r_\lambda r_\nu \Gamma_{\lambda\mu\nu}(-q-r,q,r) \;=\; r_\lambda r_\nu \Gamma_{\lambda\mu\nu}(-r,0,r) = 0\;. \tag{2.11}$$

In the usual notation and for a general tensorial decomposition of the 3-gluon vertex (see e.g. ref. [7]) this is nothing else than the known result $T_3(p^2) = 0$. Equipped with this result (which is valid, of course, when any of the arguments goes to 0) and with our previous hypothesis that $\Gamma_{\lambda\mu\nu}$ has a well defined limite when anyone of its arguments goes to 0, we can conclude that the zero-momentum limit for the l.h.s. of eq. (2.5) is null, i.e.

$$\lim_{q\to 0}\; p_\lambda r_\nu \Gamma_{\lambda\mu\nu}(p,q,r) \;=-\; r_\lambda r_\nu \Gamma_{\lambda\mu\nu}(-r,0,r) \;=\; 0\,, \tag{2.12}$$

and, of course, the same for the r.h.s. of eq. (2.5),

$$\lim_{q\to 0}\left[F(p^2)\frac{q^2}{G(q^2)}\;\left(r_\mu - \frac{(q\cdot r)}{q^2} q_\mu\right) X(q,p;r)\right] \;=\; 0\;. \tag{2.13}$$

This, in turn, implies that

$$\lim_{q\to 0} \frac{q^2}{G(q^2)} \;=\; 0\,, \tag{2.14}$$

i.e. that the gluon propagator must diverge in the infrared, unless $X(q,p;r)$ itself goes to zero. However this is certainly not the case for large enough values of p^2 when Davydychev et al's perturbative formulae ([6]) can be used.

For the Ward-Slavnov-Taylor identity to be satisfied there is a compatibility condition which *does not* involve the 3-gluon vertex (cf ref. [3]). Applying the scalar X introduced earlier in eq. (2.7) it reads

$$\frac{F(q^2)}{G(p^2)} X(p,q;r) = \frac{F(r^2)}{G(p^2)} X(p,r;q) \tag{2.15}$$

and has to be satisfied for all p's, which allows to get rid of the $G(p^2)$ denominators. Let us consider this relation in the small r limit. The X-factor on the left is the same one (except for the interchange of p and q) that appeared in the r.h.s. of eq. (2.5) and remains finite in view of the hypothesis made concerning $\Gamma_{\lambda\mu\nu}$. This implies that $F(r^2)X(p,r;q)$ too remains finite, which implies a strong correlation between the infrared behaviours of the ghost propagator and of the ghost-ghost-gluon vertex.

To summarize:

- $\Gamma_{\lambda\mu\nu}(p,q,r)$ *infrared-finite and* $X(q,p;-p-q) \neq 0 \implies \frac{G(q^2)}{q^2} \underset{q\to 0}{\longrightarrow} \infty$
- *There exists a strong relationship between IR behaviours of the ghost propagators and of the ghost-gluon vertex:* $F(r^2) \propto X(p,q;r)/X(p,r;q)$ *when* $r \to 0$ *and* $F(0)$ *is presumably finite non-zero.*

We recall that these conclusions hold for all covariant gauges. As will be briefly discussed below, the implications for gluon and ghost propagators in any particular gauge depend on the non-perturbative behaviour of the combination of scalars, $X(p,q;r)$ when q or r goes to 0. Although some perturbative results in arbitrary gauges might be invoked [6], these would be taken as nothing but a rough indication.

3. Discussion and conclusion

The Ward-Slavnov-Taylor identity, supplemented with reasonable regularity assumption for the 3-gluon vertex imposes that in any covariant gauge the gluon propagator is infrared divergent, however slowly this may be.[3] The behaviour of the dressing function in this region is usually described using an "infrared exponent":

$$G(q^2) \underset{q^2\to 0}{\widetilde{\quad}} (q^2)^{\alpha_G} \ . \tag{3.1}$$

A divergent propagator would imply either that α_G is smaller than one or that the power law is modified by logarithmic corrections. There is no such divergence on the lattice in the landau gauge: the propagator at zero momentum is obviously finite and, actually, it has been measured directly (our own simulations presently give a value of α_G close to one but

[3] In the eventuality that not only $\Gamma_{\lambda\mu\nu}(p,q,r)$ but the scalar form-factors involved in it would be finite the divergence could be still stronger, including the possibility of a divergent dressing-function.

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do not allow to exclude any of the mentioned possibilities). Therefore the divergence could only manifest itself through the volume dependence. The Adelaide group has performed a detailed analysis of this dependence (cf ref. [8]) and its results rather favours a finite IR propagator. Very recently, some authors [9] have pointed that very-low-momentum data obtained from large asymmetric lattices seem to favour a vanishing gluon propagator. However, that result is not confirmed by the authors of ref. [10, 11]. We neither agree with that conclusion and discuss about that and their implications in ref. [12]. While the WSTI and its consequences hold in all covariant gauges those observations have been made in the special case of the Landau gauge where there are of course no longitudinal gluons although the WSTI involves the non transverse vertex function; in that sense it appears as some kind of "limiting case" of the general covariant gauge. One could therefore think of a divergent term whose coefficient would vanish as $1 - \xi$ as it is the case for instance for the one-loop anomalous dimension of the ghost-gluon vertex. But this would lead to a hardly acceptable discontinuity in the r.h.s of equation (2.3) when $\xi \to 1$. A simpler and –maybe– more natural explanation would be to imagine that the rate of divergence is too slow to be seen at present.

Regarding the ghost infrared exponent, we have to make an hypothesis about the way the combination of scalars, $X(p, r; q)$ defined by eq. (2.7), in the right hand side of equation (2.15) behaves at small-r. Davydychev et al's formulae show that, at one loop, both b and d suffer from a logarithmic divergence in this limit. The one in b would be compensated by the $(r.p)$ factor in front of it, but not the one in d. In any case we do not know what the situation is for the full non-perturbative quantities. If $X(p, r; q)$ goes to some finite non-zero limit as r goes to zero $F(r^2)$ will also do. This is the situation we had considered in ref. [1, 2] and it implies that α_F is zero. If, on the contrary, the Davydychev et al's perturbative divergence persists for $X(p, r; q)$ in the non-perturbative sector, F will have to go to zero with r. The remarks we have made previously regarding the Landau gauge apply of course also for the ghost propagator but at least the finiteness of the dressing function seems to be on a safe ground since one should have to imagine a divergent part with a coefficient proportional to $\delta(1 - \xi)$.

Some of the results we have presented in this note are not new (for instance the conclusion that F(0) is finite and non zero can be found in ref. [7]) but it seems that their consequences have often been overlooked. The arguments we have presented rest exclusively on the Ward-Slavnov-Taylor identity with which, we think, any sensible solution, among which the ones derived from Schwinger-Dyson equations, should comply. In view of our own experience with lattice simulations, **our preference would go to a weakly divergent gluon propagator together with a finite and non-zero ghost dressing function**. Of course simulations with large lattices would be necessary to discriminate unambiguously between the various possibilities, as would be the use of a general covariant gauge to cope with the would-be special characteristics of the Landau gauge.

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